International Journal of Theoretical Physics, Vol. 41, No. 6, June 2002 (© 2002)

Effect of Torsion in Dirac Equation for Coulomb Potential in Robertson–Walker Space–Time

Antonio Zecca[1,2]

Received July 26, 2001

The Dirac equation with Coulomb-like potential and self-interaction term, that originates from torsion, is studied in the Robertson–Walker space–time. It is shown that the angular dependence of the equation can be separated also in presence of nonlinear terms. Under reasonable physical assumptions, the time dependence is also separated. An extended perturbative calculation can then be applied qualitatively. The conclusion is that the perturbation of the energy levels of the system, as consequence of the self-interacting term, is not relevant on physical grounds.

KEY WORDS: Dirac equation; torsion; hydrogen spectrum.

1. INTRODUCTION

The consideration of wave equations in curved space–time with torsion has always been attracting for different reasons. It allows a full description of the interaction of the particle and the gravitational field and gives new degrees of freedom to further describe physical interactions (Finkelstein, 1960; Gürsey, 1957; Hehl *et al.*, 1976; Hehl and Datta, 1971, and references therein). The interest has been finally raised in connection with the study of neutrino oscillations (Alimohammadi and Shariati, 1999; Zhang, 2000) and in general in the extensions of the Standard Model to curved space–time (Dobado and Maroto, 1996) and string theory (Hammond, 2000).

For what concerns the spin 1/2 wave equation in curved space–time with torsion, it can be obtained by the action principle by using as total action the sum of the Einstein–Hilbert–Cartan action with the Dirac action. One canonically obtains a Dirac-like equation containing nonlinear terms that are originated by the torsion (see e.g., Hehl and Datta, 1971). Recently (Zecca, 2002), the Dirac-like equation

[1] Dipartimento di Fisica dell' Universita', Via Celoria 16, I-20133 Milano, Italy.
[2] INFN, Istituto Nazionale di Fisica Nucleare, Sezione di Milano, GNFM, Gruppo Nazionale per la Fisica Matematica, Unita' di Firenze, Milano, Italy; e-mail: Zecca@mi.infn.it.

0020-7748/02/0600-1145/0 © 2002 Plenum Publishing Corporation

has been reconsidered and translated into the language of the two spinor formalism of Newman and Penrose (1962). The result leads to interpret the nonlinear terms of the equation as originated by the interaction of the particle with its own current. Besides the problem of solving the equation, it remains then to establish whether the modifications introduced by the torsion in the Dirac equation are in some way physically relevant.

The object of this paper is to test the effect of torsion on the spectrum of the Coulomb-like spinor potential. The study is performed in the Robertson–walker metric both because this metric is the base of the standard cosmology and because a similar study has been already performed in absence of torsion (e.g., Zecca, 1999). To evaluate the mentioned effect, the Dirac equation with Coulomb-like potential and nonlinear terms is preliminary separated in its angular dependence. This possibility strictly depends on the analitycal structure of the nonlinear terms and has the advantage of giving an angular dependence similar to that of the torsion free case. Under the approximation of a constant cosmological background one is then left with a pair of coupled radial equations containing nonlinear terms. By a standard manipulations of these equation, and to the first order in the nonlinear terms, it is possible to put the problem into the form of an eigenvalue-like problem. By perturbative evaluation, a qualitative effect of the torsion on the descrete spectrum of the Hydrogen-like atom is then performed. The result is such that, under the approximations done, the shifts of the energy levels due to the torsion effect is physically unappreciable in the Robertson–Walker space–time.

2. DIRAC EQUATION WITH POTENTIAL AND TORSION

The Dirac equation in spinor form in curved space–time (Chandrasekhar, 1983; Penrose and Rindler, 1984) can be formulated by including also a spinor potential (e.g., Illge, 1993; Zecca, 1999). In case of space–time with torsion, the equation is further modified by the introduction of nonlinear terms, given by the interaction of the particle with its own current, as it follows by canonically applying the action principle to a well-known Lagrangian (Zecca, 2002). By combining the results of the schemes of the mentioned papers, the Dirac equation relative to a particle of mass $m_e = \mu_\star\sqrt{2}$, subjected to a spinor Potential $V_{AA'}$ and with self-interaction induced by torsion, can be written as

$$\begin{aligned}(\nabla_{AA'} + iV_{AA'} + ibJ_{AA'})P^A + i\mu_\star\bar{Q}_{A'} = 0\\ (\nabla_{AA'} - i\bar{V}_{AA'} - ibJ_{AA'})Q^A + i\mu_\star\bar{P}_{A'} = 0\end{aligned} \tag{1}$$

(b real, $b \approx 1$) where $\nabla_{AA'}$ is the usual covariant spinor derivative and $J^{AA'} = P^A \bar{P}^{A'} + Q^A \bar{Q}^{A'}$ is the spinorial current associated to the particle. The Eq. (1) can be made explicit in terms of the directional derivatives and spin coefficients. One gets

$$\begin{aligned}
(D + \epsilon - \rho + iV_{00})P_1 - (\delta^\star + \pi - \alpha + iV_{10})P_0 + i(\mu_\star + bQ_A P^A)\bar{Q}_0 = 0 \\
(\delta + \beta - \tau + iV_{01})P_1 - (\Delta + \mu - \gamma + iV_{11})P_0 + i(\mu_\star + bQ_A P^A)\bar{Q}_1 = 0 \\
(D + \bar{\epsilon} - \bar{\rho} - i\bar{V}_{00})Q_1 - (\delta^\star + \bar{\pi} - \bar{\alpha} - i\bar{V}_{10})Q_0 + i(\mu_\star + bQ_A P^A)\bar{P}_0 = 0 \\
(\delta + \bar{\beta} - \bar{\tau} - i\bar{V}_{01})Q_1 - (\Delta + \bar{\mu} - \bar{\gamma} - i\bar{V}_{11})Q_0 + i(\mu_\star + bQ_A P^A)\bar{P}_1 = 0.
\end{aligned} \tag{2}$$

This is the general form of the Dirac equation with potential in space–time with self-interacting terms generated by torsion. Further formal developments can be obtained only by specializing the scheme. From a physical point of view it is interesting to test the effect of the torsion interacting terms (that have a gravitational origin) on the properties of some known physical system. The object of the following sections is to test this effect on the Hydrogen-like energy spectrum for a Dirac equation considered in the context of the Robertson–Walker space–time.

3. SEPARATION OF THE ANGULAR DEPENDENCE IN THE ROBERTSON–WALKER SPACE–TIME WITH TORSION

The study is hereafter confined in the space–time of metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \quad a = 0, \pm 1. \tag{3}$$

By using a null tetrad frame previously introduced, all the spin coefficients involved in (2) are real, such that $\alpha = -\beta$, $\epsilon = -\gamma$, $\tau = \pi = 0$, and do not depend on φ (Zecca, 1995, 1996). Therefore the φ dependence factors out in the form $\exp(im\varphi)$, $m = 0, \pm 1, \pm 2, \ldots$ By further setting

$$\begin{aligned}
P^A &\equiv \frac{1}{rR}\big(H_1(r,t)\exp iS_1(\theta),\, H_2(r,t)\exp iS_2(\theta)\big)\exp(im\varphi) \\
\bar{Q}^{A'} &\equiv \frac{1}{rR}\big(-H_1(r,t)\exp iS_2(\theta),\, H_2(r,t)\exp iS_1(\theta)\big)\exp(-im\varphi) \\
V_{01} &= V_{10} = 0, \qquad V_{00} = V_{00}(r), \qquad V_{11} = V_{11}(r)
\end{aligned} \tag{4}$$

(S_1, S_2 real functions) and by proceeding as in Zecca (1999), the Eq. (2) can be developed and put into the form

$$\begin{aligned}
&\sqrt{2}r\,R\frac{DH_1+(\epsilon+iV_{00})H_1}{H_2}-i\left[\mu_\star-b\frac{H_1\bar{H}_2+\bar{H}_1H_2}{(r\,R)^2}\right]\sqrt{2}r\,R\\
&\quad=-\frac{L^-\exp iS_2}{\exp iS_1}=\lambda\\
&\sqrt{2}r\,R\frac{\Delta H_2+(\epsilon+iV_{11})H_2}{H_1}-i\left[\mu_\star-b\frac{H_1\bar{H}_2+\bar{H}_1H_2}{(r\,R)^2}\right]\sqrt{2}r\,R\\
&\quad=-\frac{L^+\exp iS_1}{\exp iS_2}=-\lambda\\
&\sqrt{2}r\,R\frac{DH_1+(\epsilon+iV_{11})H_1}{H_2}-i\left[\mu_\star-b\frac{\bar{H}_1H_2+H_1\bar{H}_2}{(r\,R)^2}\right]\sqrt{2}r\,R\\
&\quad=\frac{L^+\exp iS_1}{\exp iS_2}=\lambda\\
&\sqrt{2}r\,R\frac{\Delta H_2+(\epsilon+iV_{00})H_2}{H_1}-i\left[\mu_\star-b\frac{\bar{H}_1H_2+H_1\bar{H}_2}{(r\,R)^2}\right]\sqrt{2}r\,R\\
&\quad=\frac{L^-\exp iS_2}{\exp iS_1}=-\lambda
\end{aligned}\tag{5}$$

where λ is the separation constant and it has also been set $L^\pm=\partial_\theta\mp m/\sin\theta+(\cot\theta)/2$. (One can check that, for $b=0$, the Eqs. (5) are, modulo the angular part, those found in Zecca (1999)). The angular functions then satisfy the equations $L^+\exp iS_1=-\lambda\exp iS_2$, $L^-\exp iS_2=\lambda\exp iS_1$ and also $L^-L^+\exp iS_1=-\lambda^2\exp iS_1$, $L^+L^-\exp iS_2=-\lambda^2\exp iS_2$. These last equations, that are satisfied also by the real and imaginary part of $\exp iS_1$, $\exp iS_2$, can be found solved in Montaldi and Zecca (1994). The regular solutions are of the form

$$\begin{gathered}
(1\mp\cos\theta)^{\frac{|m|}{2}+\frac{1}{4}}(1\pm\cos\theta)^{\frac{|m|}{2}-\frac{1}{4}}P(\theta),\qquad\lambda^2=\left(l+\tfrac{1}{2}\right)^2,\quad l=|m|,|m|+1,\ldots\\
(\sin\theta)^{1/2}U(\theta),\quad\lambda^2=\left(l+\tfrac{1}{2}\right)^2,\quad l=0,1,2,\ldots
\end{gathered}\tag{6}$$

for $|m|\geq 1$ and $m=0$, respectively, where P and U are real polynomials in $\cos\theta$. The expressions in (6) are bounded for $0\leq\theta\leq\pi$ and therefore, by eventually multiplying by a constant factor, the solutions $\cos S_1$, $\cos S_2$, $\sin S_1$, and $\sin S_2$ are

well known. One is then left with the equations in the r and t variables

$$\begin{aligned} DH_1 + (\epsilon + iV_{00})H_1 &= \left[i\left(\mu_\star - b\frac{H_1\bar{H}_2 + H_2\bar{H}_1}{(rR)^2}\right) - \frac{\lambda}{rR\sqrt{2}}\right] H_2 \\ \Delta H_2 + (\epsilon + iV_{11})H_2 &= \left[i\left(\mu_\star - b\frac{H_1\bar{H}_2 + H_2\bar{H}_1}{(rR)^2}\right) + \frac{\lambda}{rR\sqrt{2}}\right] H_1 \end{aligned} \tag{7}$$

where the explicit expressions of the directional derivatives and of the spin coefficients are (e.g., Zecca, 1999)

$$\begin{aligned} \sqrt{2}D &= \partial_t + \left(\sqrt{1-ar^2}/R\right)\partial_r \\ \sqrt{2}\Delta &= \partial_t - \left(\sqrt{1-ar^2}/R\right)\partial_r \\ \epsilon &= 2^{-3/2}\dot{R}/R. \end{aligned} \tag{8}$$

It is worth noticing that it has been possible to separate the angular dependence as a consequence of the particular structure of the nonlinear terms. The further separation of the r, t variables, that can be performed in the absence of potential and torsion (Zecca, 1996), is here quite difficult to be obtained. Owing to the special R dependence, the self-interaction terms can be considered small with respect to the other terms. In the following they will be treated as a small perturbation.

4. PERTURBATION OF THE HYDROGEN ENERGY SPECTRUM INDUCED BY TORSION

The object is now to evaluate the order of the perturbation of the energy levels of a hydrogen-like potential owing to the presence of the self-interacting terms. To that end, since in correspondence to the energy levels the solutions are expected to be localized into atomic dimensions and the time intervalls involved are negligible on cosmological scale, it is possible to study the equation (7) under the approximations

$$\begin{aligned} ar^2 &\ll 1 \\ R(t) &= \text{constant} = R \end{aligned} \tag{9}$$

and R sufficiently large. It is also convenient to choose instead of r, t, the independent variables

$$s = \int_0^r \frac{dr}{\sqrt{1-ar^2}}, \quad \tau = t/R. \tag{10}$$

In this way the directional derivatives become $D = (\partial_\tau + \partial_s)/R\sqrt{2}$, $\Delta = (\partial_\tau - \partial_s)/R\sqrt{2}$. The Coulomb-like potential will be represented by the

expression

$$V_{00} = V_{11} = V = \frac{1}{\sqrt{2}}\frac{1}{R}\frac{\chi}{r}. \tag{11}$$

From the positions (10), (11), by setting $rg = H_1 + H_2, rf = H_1 - H_2$, by adding and subtracting the Eqs. in (7), one obtains

$$\begin{aligned} f_s + g_\tau + \frac{r_s - \lambda}{r} f + i\sqrt{2}Rg\left[V - \mu_\star + \frac{b(g\bar{g} - f\bar{f})}{2R^2}\right] = 0 \\ f_\tau + g_s + \frac{r_s + \lambda}{r} g + i\sqrt{2}Rf\left[V + \mu_\star - \frac{b(g\bar{g} - f\bar{f})}{2R^2}\right] = 0. \end{aligned} \tag{12}$$

By the assumptions (9), $r_s \sim 1$ and the τ dependence factors out in the form $\exp(ik_o\tau)$ so that the system (12) becomes

$$\begin{aligned} f' + \frac{1-\lambda}{r} f + ig\left[\frac{\chi}{r} - m_o + k_o + \frac{b\sqrt{2}}{2R}(g\bar{g} - f\bar{f})\right] = 0 \\ g' + \frac{1+\lambda}{r} g + if\left[\frac{\chi}{r} + m_o + k_o - \frac{b\sqrt{2}}{2R}(g\bar{g} - f\bar{f})\right] = 0 \end{aligned} \tag{13}$$

where $' = d/ds$, $m_o = \sqrt{2}R\mu_\star = Rm_e$, $k_o = kR$ and f, g depend now only on the variable s. The object is now to treat the nonlinear terms as a perturbation. We set (Darwin, 1928; Gordon, 1928; Zecca, 1999)

$$\begin{aligned} f &= C\exp(-\rho/2)\rho^{\gamma-1}(q_1 + q_2) \\ g &= D\exp(-\rho/2)\rho^{\gamma-1}(q_1 - q_2) \\ \rho &= 2\delta r, \qquad \delta = R\sqrt{m_e^2 - k^2} \quad (r = r(s)) \\ C/D &= [(k - m_e)/(k + m_e)]^{1/2}. \end{aligned} \tag{14}$$

By using again the approximation $\sqrt{1 - ar^2} \cong 1$, the Eqs. (13) become, after some calculations, in terms of the functions q_1, q_2

$$\begin{aligned} \rho q_1' + q_1[\gamma + kA + m_e B] - q_2[\lambda + m_e A + kB] = 0 \\ \rho q_2' + q_2[-\rho + \gamma - kA - m_e B] + q_1[-\lambda + m_e A + kB] = 0 \end{aligned} \tag{15}$$

where now $' = d/d\rho$. Moreover in Eq. (15) it has been used the definition

$$A = \frac{\chi}{\sqrt{m_e^2 - k^2}}, \qquad B = \frac{b\sqrt{2}}{4}\frac{\rho(g\bar{g} - f\bar{f})}{(m_e^2 - k^2)R^2} \tag{16}$$

where f, g are understood to be the functions of q_1, q_2 as expressed by (14). By further setting

$$\gamma = \sqrt{\lambda^2 - \chi^2} \tag{17}$$

and neglecting terms of the order B^2, BB' one arrives at

$$\begin{aligned}
&\rho q_1'' + q_1' \left[1 + 2\gamma - \rho - \frac{\rho k B'}{\lambda + m_e A} \right] \\
&\quad - q_1 \left[\gamma + kA + m_e B - B' \frac{m_e \lambda - k\gamma + A\left(m_e^2 - k^2\right)}{\lambda + m_e A} \right] = 0 \\
&\rho q_2'' + q_2' \left[1 + 2\gamma - \rho + \frac{\rho k B'}{\lambda - m_e A} \right] \\
&\quad - q_2 \left[1 + \gamma + kA + m_e B + B' \frac{m_e \lambda - k\gamma + k\rho + A\left(m_e^2 - k^2\right)}{\lambda - m_e A} \right] = 0.
\end{aligned} \tag{18}$$

The Eq. (18) can be put into the form of an eigenvalue problem with $Q \equiv \binom{q_1}{q_2}$:

$$\begin{aligned}
&(\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2) Q = (kA + \gamma) Q \\
&\mathcal{H}_0 = \begin{pmatrix} \rho \frac{d^2}{d\rho^2} + (1 + 2\gamma - \rho) \frac{d}{d\rho} & 0 \\ 0 & \rho \frac{d^2}{d\rho^2} + (1 + 2\gamma - \rho) \frac{d}{d\rho} - 1 \end{pmatrix} \\
&\mathcal{H}_1 = \begin{pmatrix} m_e B & 0 \\ 0 & m_e B \end{pmatrix} \\
&\mathcal{H}_2 = \begin{pmatrix} -\frac{\rho k B'}{\lambda + m_e A} \frac{d}{d\rho} + B' \frac{m_e \lambda - k\gamma + A(m_e^2 - k^2)}{\lambda + m_e A} & 0 \\ 0 & \frac{\rho k B'}{\lambda - m_e A} \frac{d}{d\rho} - B' \frac{m_e \lambda - k\gamma - A(m_e^2 - k^2) + k\rho}{\lambda - m_e A} \end{pmatrix}
\end{aligned} \tag{19}$$

The zero order eigenvalue problem ($B = 0$) has been solved in Zecca (1999) and gives

$$Ak_n + \gamma = 2\gamma + n_r, \quad n_r = 1, 2, 3, \ldots \left(\gamma - \frac{\chi k_n}{\sqrt{m_e^2 - k_n^2}} = -n_r \right) \tag{20}$$

with corresponding energy levels

$$\frac{k_n}{m_e} = \left[1 + \frac{\chi^2}{\left(n_r + \sqrt{\lambda^2 - \chi^2}\right)^2} \right]^{-1/2} \tag{21}$$

that coincide with the energy levels of the hydrogen atom in flat space–time (Berestetski *et al.*, 1972; Bethe and Salpeter, 1957).

In presence of nonzero B, the terms $\mathcal{H}_1, \mathcal{H}_2$ can be treated as small perturbations. The correction ζ of the energy levels (20) induced by $\mathcal{H}_1, \mathcal{H}_2$ could then be calculated at a first approximation, by first order perturbation theory, that is by calculating expressions like $\langle Q_n|\mathcal{H}_1 + \mathcal{H}_2|Q_n\rangle$. The Q_n are the zero-order eigenfunctions and $\mathcal{H}_1, \mathcal{H}_2$ are understood to be themselves expressed in (19) by the zero-order eigenfunctions by using the definitions (16) and (14) [Similar point of view was adopted in Cavalleri and Zecca (1991) for perturbative calculation relative to a nonlinear Schrödinger equation].

The corrections k'_n to the energy levels satisfy the equation

$$2\gamma + n_r + \zeta = \gamma + \frac{\chi k}{\sqrt{m_e^2 - k^2}} \approx 2\gamma + n_r + k'_n \frac{\chi m_e^2}{\left(m_e^2 - k_n^2\right)^{3/2}} \tag{22}$$

where also Eq. (21) has been used. Therefore

$$k'_n \approx \zeta \frac{\left(m_e^2 - k_n^2\right)^{3/2}}{\chi m_e^2} \approx \frac{k_n^3 \chi^2}{m_e^2} \zeta \tag{23}$$

and ζ can be obtained as mentioned. Since we are interested only in the order of the magnitude of the perturbation and since the dominant term in the perturbation is $\sim m_e B$, one has

$$\zeta \approx \frac{m_e}{R^2\left(m_e^2 - k_n^2\right)} \approx \frac{m_e}{k_n^2 \chi^2 R^2}. \tag{24}$$

Therefore the relative perturbation of the levels is

$$\frac{k'_n}{k_n} \approx \frac{k_n}{m_e^2 R^2} \approx \frac{1}{m_e R^2}. \tag{25}$$

By passing from Plank units to ordinary units one has $k'_n/k_n \simeq l_p^3 c^2/(G m_e R^2) \simeq 10^{-43}/R^2$. To have k'_n/k_n at least of the order of the relative Lamb shift, that is of the order $\alpha^3 = (1/137)^3$, it must be $R \leq 10^{-18}$.

5. CONCLUDING REMARKS

The results of the previous perturbative calculations are valid under the second condition (9) and are therefore exact for $a = 0$ and R constant and arbitrary. In general, as pointed out by Parker (1980), there is a shift of the energy levels, because of gravitational field, proportional to a curvature-like parameter D. In case of Robertson–Walker space–time models the parameter is of the form $D \sim 1/R$. Accordingly for $1/R \leq 10^{-4}$ cm^{-1} the spectrum (21) is right in the sense that its modification due to the gravitational field is of an order of magnitude less then the

Lamb shift of the Hydrogen atom (Parker, 1980). Therefore, under the condition $R > 10^4$ the spectrum (21) can be correctly considered as the zero-order spectrum of the previous calculations. The conclusion of the previous section is then that the effect of the self-interaction term to the energy levels is so small to be practically unobservable.

If one now considers values of R for which the gravitational effect on the energy levels is at least of the order of the Lamb shift, say $R < 10^3$, the previous perturbative calculation could be repeated in principle, if one is able to find an analytic expression for the energy levels.

What seems to be more problematic, in this case, is however the validity of the second condition in (9). By relaxing this condition, the separation of r, t variables, that is possible in absence of self-interacting terms and potential (Zecca, 1996), becomes very difficult. Also the applicableness of the perturbative method would be quite questionable in this case. A complete solution in this sense would be, of course, very satisfactory. The guess is, however, that if the previous calculations are developed in the context of the standard cosmological model (e.g., Kolb and Turner, 1990), then to have physically measurable modification of the energy spectrum, one has to go into epochs where the existence of the Hydrogen atom itself is questionable.

REFERENCES

Alimohammadi, M. and Shariati, A. (1999). Neutrino oscillation in space-time with torsion. *Modern Physics Letters A* **14**, 267.

Berestetski, V., Lifchitz, E., and Pitayevski, L. (1972). *Theórie Quantique Relativiste*, Part 1, Editions MIR, Moscow.

Bethe, H. A. and Salpeter, E. E. (1957). *Quantum Mechanics of One- and Two-Electron Atoms*, Springer-Verlag, Berlin.

Cavalleri, G. and Zecca, A. (1991). Interpretation of a Schrödinger-like equation derived from a non-Markovian process. *Physical Review B* **43**, 3223.

Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, Oxford University Press, New York.

Darwin, G. G. (1928). *Proceedings of the Royal Society of London A* **118**, 654.

Dobado, A. and Maroto, A. L. (1996). The standard model anomalies in curved space-time with torsion. *Physical Review D: Particles and Feilds* **54**, 5185.

Finkelstein, R. (1960). Spacetime of elementary particles. *Journal of Mathematical Physics* **1**, 440.

Gordon, W. (1928). Die Energieniveaus des Wasserstoffatom nach der Dirac schen Quantentheorie des Elektrons. *Zeitschrift für Physik* **48**, 11.

Gürsey, F. (1957). General relativistic interpretation of some spinor wave equations. *Il Nuovo Cimento* **5**, 154.

Hammond, R. T. (2000). Strings in gravity with torsion. *General Relativity and Gravitation* **32**, 2007.

Hehl, F. W. and Datta, B. K. (1971). Nonlinear spinor equation and asymmetric connection in general relativity. *Journal of Mathematical Physics* **12**, 1334.

Hehl, F. W., von der Heyde, P., Kerlick, G. D., and Nester, J. M. (1976). General relativity with spin and torsion: Foundations and prospects. *Review of Modern Physics* **48**, 393.

Illge, R. (1993). Massive fields of arbitrary spin in curved space-times. *Communications in Mathematical Physics* **158**, 433.

Kolb, E. W. and Turner, M. S. (1990). *The Early Universe*, Addison-Wesley, New York.

Montaldi, E. and Zecca, A. (1994). Neutrino wave equation in the Robbertson–Walker geometry. *International Journal of Theoretical Physics* **33**, 1053.

Newmann, E. T. and Penrose, R. (1962). An approach to gravitational radiation by a method of spin coefficients. *Journal of Mathematical Physics* **3**, 566.

Parker, L. (1980). One-electron atom as a probe of space-time curvature. *Physical Review D*: Particles and Feilds **22**, 1922.

Penrose, R. and Rindler, W. (1984). *Spinors and Space-time*, Vols. I, II, Cambridge University Press, Cambridge.

Zecca, A. (1995). Non conservation of the net current of dirac particles in Toman–Bondi and Robertson–Walker geometry. *International Journal of Theoretical Physics* **34**, 1083.

Zecca, A. (1996). The dirac equation in the Robertson–Walker space-time. *Journal of Mathematical Physics* **37**, 874.

Zecca, A. (1999). Dirac equation with electromagnetic potential: Discrete spectrum of the hydrogen atom in the Robertson–Walker space-time. *International Journal of Theoretical Physics* **38**, 945.

Zecca, A. (2002). Dirac equation in space-time with torsion. *International Journal of Theoretical Physics* **41**, 421.

Zhang, C. M. (2000). Mass neutrino flavor evolution in spacetime with torsion. *Il Nuovo Cimento B* **115**, 437.